Approximation for Unsteady Aerodynamics," *Journal of Aircraft*, Vol. 28, No. 9, 1991, pp. 545–552.

[8]Tewari, A., "New Acceleration Potential Method for Supersonic Unsteady Aerodynamics of Lifting Surfaces, Further Extension of the Nonplanar Doublet Point Method, and Nonlinear, Nongradient Optimized Rational Function Approx. for Supersonic, Transient Response Unsteady Aerodynamics," Ph.D. Dissertation, Univ. of Missouri—Rolla, Rolla, MO, 1992.

[9]Tewari, A., and Brink-Spalink, J., "Multiple Pole Rational-Function Approximations for Unsteady Aerodynamics," *Journal of Aircraft*, Vol. 30, No. 3, 1993, pp. 426–428.

[10]Dowell, E. H., Bland, S. R., and Williams, M. H., "Linear/Nonlinear Behavior in Unsteady Transonic Aerodynamics," *AIAA Journal*, Vol. 21, No. 1, 1983, pp. 38–46.

[11]Albano, E., and Rodden, W. P., "A Doublet-Lattice Method for Calculating the Lift Distribution on Oscillating Surfaces in Subsonic Flow," *AIAA Journal*, Vol. 7, No. 2, 1969, pp. 279–285.

[12]Ueda, T., and Dowell, E. H., "Doublet-Point Method for Supersonic Unsteady Lifting-Surfaces," *AIAA Journal*, Vol. 22, No. 2, 1984, pp. 179–186.

Fractional Order Calculus Model of the Generalized Theodorsen Function

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Introduction

THE generalized Theodorsen function relates unsteady, circulatory lift to unsteady circulation for small perturbation motion of a flat plate in two-dimensional, incompressible, inviscid flow.[1] Because the generalized Theodorsen function is based on first principles, it is an appropriate bench mark for modeling unsteady aerodynamic forces on thin airfoils. The generalized Theodorsen function is a Laplace transform composed of tanscendental functions and it has no closed-form inverse transform. These characteristics make the generalized Theodorsen function somewhat cumbersome to manipulate in analyses. This motivates the development of algebraic approximations, having inverse transforms, that are suitable for use in aeroelastic stability analyses, gust response predictions, and control system design.

It is possible to construct an algebraic, global s-plane approximation of the generalized Theodorsen function using fractional powers of s. The approximation is mathematically compact and has a known, but unfamiliar, inverse transform that is also compact. The approximation also leads to equations of motion equally well-posed in either the Laplace domain or the time domain.[2]

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Background

The generalized Theodorsen function $C(s)$ is a combination of Hankel functions of the first kind, H_0 and H_1:

$$C(s) = \frac{H_1(\bar{s})}{H_0(\bar{s}) + H_1(\bar{s})}, \qquad \bar{s} = \frac{sb}{U} \tag{1}$$

Here, $\bar{s}$ is a dimensionless form of the Laplace parameter, where b is the flat plate airfoil's semichord, and U is the freestream velocity. The Theodorsen function relates the plate's circulatory lift to the circulation of the flow about the plate

$$L(s) = \rho U \Gamma(s) C(s) \tag{2}$$

where $L(s)$ and $\Gamma(s)$ are the Laplace transforms of the time-dependent circulatory lift and the time-dependent circulation, respectively.

Unfortunately, the Theodorsen function has no known, closed-form representation of its inverse transform. This difficulty, coupled with the transcendental nature of the Hankel functions, has motivated the formulation of algebraic models of Theodorsen's function.

Commonly encountered algebraic models take the form

$$C(s) \approx \frac{N(s)}{D(s)} = \frac{\sum_{n=0}^{N} a_n \bar{s}^n}{\sum_{n=0}^{N} b_n \bar{s}^n} \tag{3}$$

where $N(s)$ and $D(s)$ are polynomial functions of the Laplace parameter $\bar{s}$. These models have inverse transforms that are a unit impulse function plus a combination of elementary analytic functions, typically exponentials.

A drawback of these models is their poles that arise from the roots of $D(s)$. Theodorsen's function has no poles in the s plane. Consequently, the poles in the models are sources of substantial, but localized, inaccuracies that detract from the model's fidelity. When one of these models is used to describe aerodynamic forces in equations of motion for an airfoil, attention must be devoted to keeping the poles introduced by the model well away from poles associated with airfoil motion. As a result, different models might be needed for flutter analysis, divergence analysis, and control system design, respectively.

Fractional Calculus Model

The primary motive for introducing concepts from the fractional order calculus is to construct a single, high-fidelity, global s-plane model of Theodorsen's function suitable for all the above analyses. This new model employs the notion of differentiating to fractional order. The extended Riemann-Liouville definition[3] for the fractional β order derivative of the function $x(t)$ is

$$D^{\beta}[x(t)] \equiv \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x(t-\tau)}{\tau^{\beta}} \, \mathrm{d}\tau \qquad 0 < \beta < 1 \tag{4}$$

Where Γ denotes the gamma function instead of circulation. The Laplace transform of Eq. (4) reveals the operative property of fractional order differentiation:

$$\mathscr{L}\{D^{\beta}[x(t)]\} = s^{\beta}\mathscr{L}\{x(t)\} \tag{5}$$

The fractional calculus model of Theodorsen's function, $\hat{C}(s)$, shown here, contains fractional powers of $\bar{s}$:

$$\hat{C}(s) = \frac{1 + (2.19)\bar{s}^{5/6}}{1 + 2(2.19)\bar{s}^{5/6}}, \qquad \bar{s} = \frac{sb}{U} \tag{6}$$

This model is based on a visual "fitting" of the model to the function along the imaginary $\bar{s}$ axis. This model is compared to the actual Theodorsen's function in Figs. 1 and 2. Note that Fig. 2 shows the negative imaginary part of the function and its model. Although not immediately apparent in Figs. 1 and 2, the real and imaginary asymptotes of Theodorsen's function and the fractional calculus model match for large magnitude of $\bar{s}$; in that, the real asymptote is one-half and the imaginary asymptote is zero.

The model, like Theodorsen's function, has no poles in the $\bar{s}$ plane. The poles of the fractional calculus model are

$$\bar{s} = \left[\frac{-1}{2(2.19)}\right]^{6/5} = \frac{\exp\left(\pm\frac{i6\pi}{5}\right)}{(4.38)^{6/5}} \tag{7}$$

These poles lie at angles of $\pm 6\pi/5$ rad measured from the positive, real $\bar{s}$ axis. When the branch cuts for the function $\bar{s}^{5/6}$ are chosen along the negative real axis, $\bar{s} = re^{i\pi}$ and $\bar{s} = re^{-i\pi}$, the model's poles given above are not on the principal sheet of the $\bar{s}$ plane. The branch cuts restrict the domain of a function to ensure that the function remains single-valued. Thus, the poles of the model lie beyond the branch cuts on nonprincipal branches of the function $\bar{s}^{5/6}$. Notice that in Fig. 2 the model's imaginary curve, for $\theta = 180$ deg or π radians, has peaks that indicate the presence of a singularity near, but just beyond the branch cut.

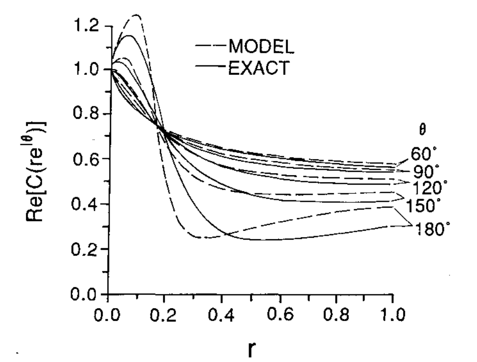

Fig. 1 Real part of Theodorsen's function and its fractional calculus model.

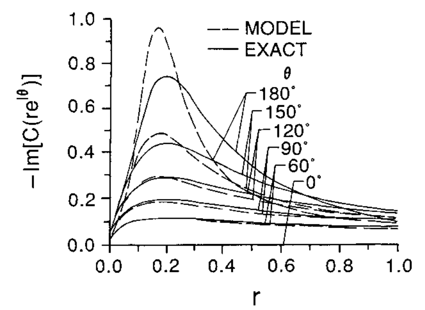

Fig. 2 Imaginary part of Theodorsen's function and its fractional calculus model.

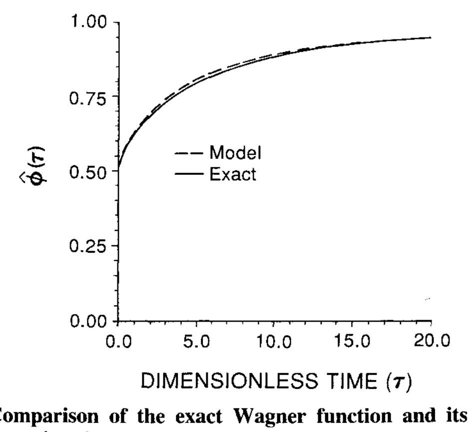

Fig. 3 Comparison of the exact Wagner function and its Mittag-Leffler approximation.

The fidelity of the fractional calculus model to the Theodorsen function shown in Figs. 1 and 2 establishes its accuracy at lower reduced frequencies and correspondingly longer times. To check the model for higher reduced frequencies and correspondingly shorter times, the Wagner's function $\hat{\phi}(\tau)$ that arises from the fractional calculus model is determined and compared with the actual Wagner's function[4] in Fig. 3.

$$\hat{\phi}(\tau) = \mathcal{L}^{-1}\left[\frac{\hat{C}(s)}{\bar{s}}\right] = u_{-1}(\tau) - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left[\frac{\tau^{5/6}}{2(2.19)}\right]^{n}}{\Gamma(1 + 5n/6)}$$

$$\tau = \frac{tU}{b} \tag{8}$$

The sum here results from a geometric series expansion of the Theodorsen function's model $\hat{C}(s)$, and yields the series representation of a Mittag-Leffler function.[5] The excellent agreement between the Mittag-Leffler approximation and the exact Wagner's function shown in Fig. 3 establishes the accuracy of the fractional calculus model of Theodorsen's function at high reduced frequencies. Furthermore, linear differential equations containing fractional derivatives give rise to solutions posed in terms of time-dependent Mittag-Leffler functions.[2]

In summary, the fractional calculus model is seen to be an excellent algebraic approximation of the transcendental Theodorsen's function. The model exhibits high fidelity throughout the $\bar{s}$ plane, and consequently, yields excellent accuracy in the time domain.

Example Problem

The fractional calculus model of Theodorsen's function is used to perform a stability analysis of the spring suspended, flat plate airfoil subjected to uniform, incompressible, inviscid airflow, considered in Ref. 6. The plate model used in the "exact" analysis contains a trailing-edge flap having a very high resonance that does not significantly affect the pitch or plunge stability of the plate.

Rather than calculating only the critical flow speeds, the stability of the plate is examined for all flow speeds up to the critical speed associated with the first instability. This is accomplished by tracking the plate's poles in the complex s plane and noting the flow speed at which any pole migrates across the imaginary axis from left to right. As the flow speed in-

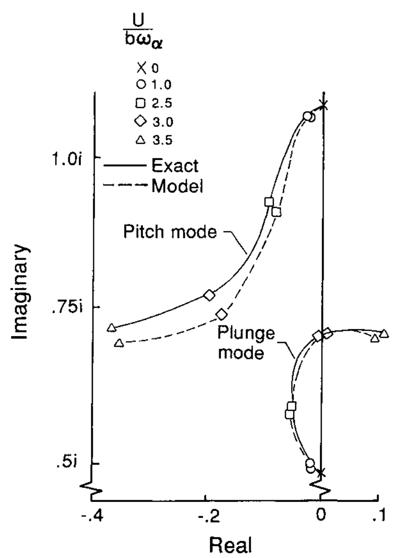

Fig. 4 s-Plane root loci for the suspended flat plate airfoil.

creases, a pole location can be tracked on a root-locus in the complex s plane as shown in Fig. 4. When the first critical flow speed is reached, the root locus shows the pole on the s-plane imaginary axis.

The plate's parameters have the following values.[6] The aerodynamic center and the location of the elastic axis is $a = -0.4 \cdot b$. The mass center location is $x_\alpha = 0.2 \cdot b$. The plate's radius of gyration is $r_\alpha = 0.5 \cdot b$. The mass ratio is $\mu = 40$ and the semichord $b = 1$. The pure pitch and plunge resonances are $\omega_\alpha = 100$ rad/s and $\omega_h = 50$ rad/s, respectively.

Conclusion

The fractional calculus model overcomes much of the impractical nature of Theodorsen's function. The model is an accurate algebraic description of the transcendental Theodorsen's function and it leads to an excellent, closed form approximation of Wagner's function. Furthermore, the fractional calculus model has no poles on the principle sheet of the $\bar{s}$ plane that obviates the difficulties associated with the aerodynamic poles associated with traditional algebraic models of Theodorsen's function. These strengths enable the model to conveniently describe the effects of unsteady aerodynamic loads for general small perturbation airfoil motion in both the Laplace domain and the time domain.

References

[1]Edwards, J. W., Ashley, H., and Breakwell, J. V., "Unsteady Aerodynamic Modeling for Arbitrary Motion," *Proceedings of the 18th Structures, Structural Dynamics and Materials Conference*, March 21–23, 1977 and Dynamics Specialists Conf., March 24–25, 1977, San Diego, CA, *Technical Papers*, Vol. B, 1977, pp. 267–283 (AIAA Paper 77-451).

[2]Bagley, R. L., and Calico, R. A., "Fractional Order State Equations for the Control of Viscoelastically Damped Structures," *Journal of Guidance Control and Dynamics*, Vol. 14, No. 2, 1991, pp. 304–311.

[3]Oldham, K. B., and Spanier, J., *The Fractional Calculus*, Academic Press, Orlando, FL, 1974.

[4]Sears, W. R., *Collected Papers of W. R. Sears Through 1973*, Cornell Univ. Press, Ithaca, NY, 1973, p. 38.

[5]Mittag-Leffler, G., "Sur La Represèntation Analytique D'une Branch Uniforme D'une Function Monagine," *Acta Mathematica*, Vol. 29, 1905, p. 101.

[6]Edwards, J. W., "Unsteady Aerodynamic Modeling for Arbitrary Motions," *AIAA Journal*, Vol. 15, No. 4, 1977, pp. 593–595.

Downwash Measurements on a Pitching Canard-Wing Configuration

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Introduction

MUCH of the experimental work associated with unsteady flows has centered around the harmonic oscillations of a single lifting surface, usually rotated about the quarter chord, and the resulting changes in the flowfield surrounding the model.[1,2] Only a few investigations[3,4] have considered the translation and rotation of finite wings such as those encountered on real, full-size, fighter-type, superma-neuverable aircraft in deep stall. Probably the best summary of the state of the art using a theoretical approach is found in Refs. 5–7. In Ref. 5, a structured approach to the problems associated with a theoretical analysis of oscillating airfoils is provided in which the various terms in the force and moment equations are discussed along with a two-dimensional solution for simple harmonic motion. For the examples chosen in the analysis, prediction of phase angles, due to circulation lag, of about 15 deg for specific configurations seems to be typical. It was shown in Ref. 8 that the maximum lift coefficient experienced during sinusoidal oscillation changes drastically with alterations of the lifting surface rotation point. For the two-wing case, with the rotation point somewhere between the two wings, one would expect significant deviations from the quarter chord rotation point data. In a two-dimensional experiment by Walker,[9] a forward airfoil was pitched about its quarter chord axis at a constant rate. Flow visualization results indicated that the leading-edge vortex, separated from the forward airfoil section, could be made to pass either over or under an aft airfoil. In this tandem wing arrangement, such as those described in Ref. 9, the downwash loading from forward lifting surfaces can make significant contributions to the loading on a downstream wing or tail. Because downwash loads and moments are difficult to predict theoretically, especially for unsteady motions, a series of wind-tunnel tests were completed in a low-speed wind tunnel on a generic configuration as pictured in Fig. 1, in which attempts were made to measure downwash directly.[10]

The experimental wind-tunnel tests were conducted in a low-speed, open return wind tunnel. The test section was 3 × 3 ft (91.44 × 91.44 cm), and the nominal freestream velocity was 89 ft/s, producing a Reynolds number of about 2.05 × 10^5 based on the wing chord length of 6.0 in. All airfoil sections were NACA 0015, and the wings were mounted to individual load and moment balances at the quarter chord of each wing. The two wing assemblies, as shown schematically in Fig. 1, were mounted to an oscillating splitter plate connected to a shaft and bearing assembly in the floor of the wind-tunnel test section. The wings were mounted in the vertical direction for ease of construction and ready access to the drive assembly underneath the tunnel floor. The round splitter plate was beveled to a 30-deg angle to reduce boundary layer and hardware interference in the test data. Each wing was mounted to a shaft along the quarter chord which was rigidly attached to individual load and moment balances. This shaft, running through the quarter chord, also served as a means of setting each wing incidence angle. The bottom of each shaft was fixed to a rigid plate bolted directly to the face of each balance, and the plate was designed so that its upper surface was flush with the splitter plate surface. A small gap between the root or bottom end of the wing raised the wing to near the edge of the boundary layer on the oscillating plate.

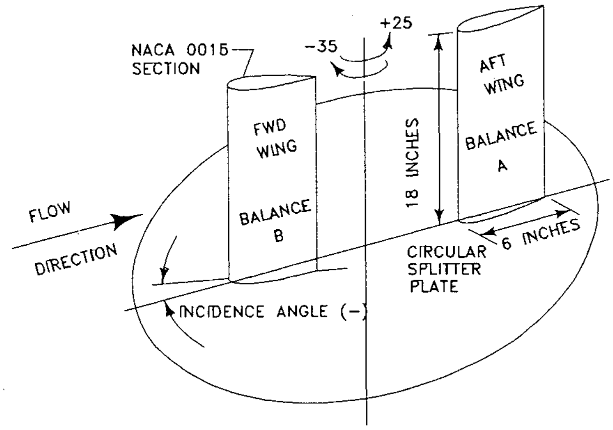

Fig. 1 Schematic of splitter plate model.

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